SACLANTCEN MEMORANDUM
serial no.: SM-295

SACLANT UNDERSEA
RESEARCH CENTRE
MEMORANDUM

ANALYSIS OF A SIGNAL STARTING TIME ESTIMATOR BASED ON THE PAGE TEST STATISTIC

D.A. Abraham

December 1995

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SACLANT Undersea Research Centre
Viale San Bartolomeo 400
19138 San Bartolomeo (SP), Italy

tel: +39-187-540.111
fax: +39-187-524.600

e-mail: library@saclantc.nato.int

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David L. Bradley
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Analysis of a signal starting time estimator based on the Page test statistic

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Executive Summary: In the detection, localization, and classification of targets in anti-submarine warfare using active sonar, the presence of a target echo must first be detected in the received array data. From the information obtained by the detection, the target is marginally localized in range by the time of detection and in bearing by implementing detectors on multiple beams that selectively accept energy from specific directions. The target echo must then be segmented in the received array data prior to refining the initial bearing and range estimates, depth estimation, or target classification. This requires estimation of the start and stop time of the target echo, a particularly difficult problem when the effect on the transmitted signal due to propagation through the ocean medium and reflection off the target are unknown prior to localization and classification.

In this memorandum a starting (or stopping) time estimator based on the Page test statistic is proposed and analyzed. The Page test is a sequential test used to detect a change; in this case the transition from signal-absence to signal-presence. The effects of signal power and the Page test detection threshold are investigated for two signal models. The results of this analysis may be used to ensure, within some probability, that the target echo segmentation includes the complete echo.

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Analysis of a signal starting time estimator based on the Page test statistic

D. A. Abraham

Abstract: The time of the most recent reset to zero of the Page test statistic is proposed as an estimator of the starting time of a signal. The probability mass function of the estimator is determined analytically subject to a quantization of the Page test statistic update. Closed form results for the first three uncorrected moments of the estimator are presented. The analytical results are verified by comparison with simulation results and the fineness of the quantization required for accurate representation is investigated by evaluation of the Kolmogorov-Smirnov statistic. The bias, standard deviation, and skewness of the estimator as a function of the signal strength and detector threshold are evaluated for Gaussian shift-in-mean and noncentral chi-squared signal types.

Keywords: Page test ∘ submarine detection, localization, classification ∘ signal starting time estimator

Contents

1
Introduction

Many signal processing applications require the detection of a change occurring in a sequence of observed data [1]. One type of application may be represented by a model where the time of occurrence of the signal is unknown and once the signal starts, it does not stop until appropriate action has been taken. Examples of this type include the detection of mechanical failures in machinery, the detection of changes in heart sounds, and the rapid detection of approaching ships or torpedoes in sonar applications. Another type of change detection may be represented by a signal model where the duration of the change is finite and both the duration and time of occurrence are unknown. Examples of this type include the segmentation of words in speech processing and the detection of active sonar echoes in shallow water. In most of these examples, once the signal is detected some action is taken that requires an estimate of when the signal started (and possibly when it has stopped). For instance, the components produced by a machine may need to be examined for adherence to design specification once a failure has occurred, a word may need to be classified once it has been segmented in speech, or an active sonar signal echo may be localized in depth or classified once it has been detected and segmented.

To mathematically introduce the change detection problem, suppose that the data X_1, X_2, ... are observed. For time indices $i \le n_s$ assume the data is distributed according to a null or before-change rule (H_0) and for $i > n_s$ assume the data is distributed according to an alternative or after-change rule (H_1). It is desired to determine when the change has occurred with as little data as possible and to subsequently provide an estimate of the time of change, n_s.

The Page test [2] has long been used to determine when the distribution of a sequence of random variables has changed. The recursive form of the test declares that a change has occurred when the statistic

$$W_k = \max\{0, W_{k-1} + g(X_k)\} \tag{1}$$

crosses a threshold where $W_0 = 0$ and $g(\cdot)$ is the Page test update or nonlinearity. The Page test may also be viewed as consecutive Wald sequential probability ratio tests (SPRTs) terminating at H_0 followed by one terminating at H_1. From this interpretation of the Page test, a natural estimator of the starting time of the signal is the time of the most recent SPRT termination at H_0; that is, the time of the

most recent reset of W_k to zero. Mathematically the starting time estimator may be represented by

$$M = \max\{k < N : \ W_k = 0\}, \tag{2}$$

where N is the stopping time of the Page test,

$$N = \inf\{k > 0 : \ W_k \geq h\}. \tag{3}$$

A direct application of the maximum likelihood estimation principle to this problem is not possible unless the change has already been detected resulting in a fixed sample size estimation problem. Under this condition, the proposed estimator is shown to be related to the maximum likelihood estimator (MLE). First, define the cumulative summation of the Page test update as

$$S_k = \sum_{i=1}^{k} g(X_i). \tag{4}$$

Now the Page test may be described non-recursively as

$$\begin{aligned} W_k &= S_k - \min_{i \leq k} S_i \\ &= \max_{i \leq k} \sum_{j=i+1}^{k} g(X_j). \end{aligned} \tag{5}$$

From this description of the Page test, the proposed starting time estimator is

$$M = \arg\max_{i \leq N} \sum_{j=i+1}^{N} g(X_j). \tag{6}$$

If the Page test update $g(x)$ is the log-likelihood ratio, then Eq. (6) has the same form as the MLE of the signal starting time except that here the sample size N is a random variable determined by the termination of the Page test.

For the detection and segmentation of finite duration signals one may use consecutive Page tests with alternating hypotheses [3]. In this case, the stopping time of the signal is similarly estimated using the time of the most recent SPRT termination at H_1.

This paper develops a method for determining the probability mass function (PMF) of the above estimator by first quantizing the Page test update $g(x)$ as shown in Sect. 2. Equations are presented for the first three moments from which the variance and skewness may be determined. In Sect. 3 the analytical results are verified by comparing the PMF and cumulative distribution function (CDF) with simulation results. Additionally, the Kolmogorov-Smirnov statistic is determined as a function

of the fineness of the quantization of the Page test update. Finally, in Sect. 4 the performance of the estimator is determined as a function of the threshold used in the Page test and the signal strength for Gaussian shift-in-mean and noncentral chi-squared signal types.

2
Estimator analysis

2.1 Quantization

Following the method of Brook [4] and as further described in reference [5], the Page test update $g(x)$ is quantized into levels equally spaced at intervals of width Δ. Arbitrarily, let the levels be $l_i = i\Delta$ for $i = 0, \pm1, \pm2, \ldots$. Let the quantization rule be $\mathcal{Q}(g) = l_i$ if $g \in [\Delta(i-0.5), \Delta(i+0.5))$, where $\mathcal{Q}(\cdot)$ represents the quantization operation. Let the integer γ be such that $l_{\gamma-1} < h \le l_\gamma$ where h is the threshold for the Page test. Thus, while the Page test is running (before a threshold crossing), the quantized statistic must lie in one of the *states* $\{l_i\}_{i=0}^{\gamma-1}$.

Suppose that under H_j for $j = 0$ or 1 the Page test update has probability distribution function (PDF) $f_j(g)$ and CDF $F_j(g)$. Then the probabilities of observing the levels under H_0 are

$$\begin{aligned} p_i &= \Pr_0\{\Delta(i-0.5) \le g < \Delta(i+0.5)\} \\ &= F_0(\Delta(i+0.5)) - F_0(\Delta(i-0.5)) \end{aligned} \tag{7}$$

and under H_1 are

$$\begin{aligned} q_i &= \Pr_1\{\Delta(i-0.5) \le g < \Delta(i+0.5)\} \\ &= F_1(\Delta(i+0.5)) - F_1(\Delta(i-0.5)). \end{aligned} \tag{8}$$

2.2 Probability mass function

There are two possible scenarios that must be considered separately in determining the PMF of the starting time estimator M: (i) when the Page test statistic does not reset to zero before crossing the threshold once the signal starts, and (ii) when it does reset to zero before crossing the threshold. Without a loss of generality the actual signal start time can be subtracted from all time indices and the starting time estimator M so only the estimator error is considered. Case (i) then represents negative values of M and case (ii) positive values of M. Note that the data now follows H_0 for times $k \le 0$ and follows H_1 for times $k > 0$.

Let the variable $\mathbf{u}$ represent a vector of length γ composed of the probability of being in states $l_0, l_1, \ldots, l_{\gamma-1}$. Let the variable $\mathbf{v}$ represent a vector of length $\gamma - 1$

composed of the probability of being in states l_1, l_2, ..., $l_{\gamma-1}$. These variables will be appended with subscripts to distinguish various conditions or time dependence. The $\mathbf{u}$ variables will be referred to as state probability vectors and the $\mathbf{v}$ variables as *reduced* state probability vectors. Also note that they may be related by the matrix-vector product

$$\begin{aligned} \mathbf{v} &= [\mathbf{0} \ \ \mathbf{I}_{\gamma-1}]\,\mathbf{u} \\ &= \mathbf{P}\mathbf{u}, \end{aligned} \tag{9}$$

where $\mathbf{P}$ is a $\gamma-1$ by γ matrix composed of a length $\gamma-1$ vector of zeros ($\mathbf{0}$) and a dimension $\gamma-1$ identity matrix ($\mathbf{I}_{\gamma-1}$).

Let $\mathcal{S}(\mathbf{u}_0)$ represent the event that the Page test statistic proceeds directly to the threshold without entering the state l_0 when the state probability vector is initially $\mathbf{u}_0$ and the data follows condition H_1 (i.e., the change has occurred). In this situation, state l_0 may only be observed at the initial time sample, say $k=0$. Thus, for all times $k>1$, the reduced state probability vector may be related to the most previous one by

$$\mathbf{v}_k = \mathbf{D}_1\mathbf{v}_{k-1}, \tag{10}$$

where $\mathbf{v}_k$ is the reduced state probability vector for time sample k and $\mathbf{D}_1$ is a $\gamma-1$ by $\gamma-1$ matrix of the probability of a transition from one state to another under H_1,

$$\mathbf{D}_1 = \begin{bmatrix} q_0 & q_{-1} & \cdots & q_{-\gamma+2} \\ q_1 & q_0 & \cdots & q_{-\gamma+3} \\ \vdots & \vdots & \ddots & \vdots \\ q_{\gamma-2} & q_{\gamma-3} & \cdots & q_0 \end{bmatrix}. \tag{11}$$

The length γ state probability vector for time $k=0$, $\mathbf{u}_0$, may be partitioned into

$$\mathbf{u}_0 = \begin{bmatrix} u_a \\ \mathbf{v}_0 \end{bmatrix}, \tag{12}$$

where $u_a = \mathbf{e}_0^T\mathbf{u}_0$ is the probability of being in state l_0, $\mathbf{e}_0 = [1\ 0\ \cdots\ 0]^T$ is a dimension γ or $\gamma-1$ unit vector pointing toward the first coordinate, and $\mathbf{v}_0 = \mathbf{P}\mathbf{u}_0$. Then, the reduced state probability vector for time $k=1$ is

$$\begin{aligned} \mathbf{v}_1 &= u_a\mathbf{d}_1 + \mathbf{D}_1\mathbf{v}_0 \\ &= [\mathbf{d}_1 \ \ \mathbf{D}_1]\,\mathbf{u}_0 \\ &= \tilde{\mathbf{D}}_1\mathbf{u}_0, \end{aligned} \tag{13}$$

where $\tilde{\mathbf{D}}_1 = [\mathbf{d}_1 \ \ \mathbf{D}_1]$, $\mathbf{d}_1$ is a vector composed of the probability of a transition from state l_0 to the states l_1, ..., $l_{\gamma-1}$ under H_1,

$$\mathbf{d}_1 = [q_1\ q_2\ \cdots\ q_{\gamma-1}]^T. \tag{14}$$

Now the probability of a straight climb to the threshold, without entering state l_0, is the probability of reaching or exceeding the threshold on the next update from each state times the probability of being in each state for each future time sample,

$$
\begin{aligned}
\Pr\{\mathcal{S}(\mathbf{u}_0)\} &= c_a u_a + \mathbf{c}^T\mathbf{v}_0 + \sum_{k=1}^{\infty} \mathbf{c}^T\mathbf{v}_k \\
&= c_a u_a + \mathbf{c}^T\mathbf{v}_0 + \mathbf{c}^T \sum_{k=1}^{\infty} \mathbf{D}_1^{k-1}\mathbf{v}_1 \\
&= c_a u_a + \mathbf{c}^T\mathbf{v}_0 + \mathbf{c}^T(\mathbf{I}_{\gamma-1} - \mathbf{D}_1)^{-1}\tilde{\mathbf{D}}_1\mathbf{u}_0 \\
&= \mathbf{w}^T\mathbf{u}_0, \qquad (15)
\end{aligned}
$$

where $c_a = \sum_{i=\gamma}^{\infty} q_i$,

$$
\mathbf{c} = \begin{bmatrix} \sum_{i=\gamma-1}^{\infty} q_i \\ \sum_{i=\gamma-2}^{\infty} q_i \\ \vdots \\ \sum_{i=1}^{\infty} q_i \end{bmatrix} \qquad (16)
$$

and

$$
\mathbf{w} = \begin{bmatrix} c_a \\ \mathbf{c} \end{bmatrix} + \tilde{\mathbf{D}}_1^T(\mathbf{I}_{\gamma-1} - \mathbf{D}_1)^{-T}\mathbf{c}. \qquad (17)
$$

Note that the development of Eq. (15) requires that the eigenvalue with the maximum absolute value of $\mathbf{D}_1$ be less than one. This property will be established in Sect. 2.3.

Now consider case (i) where, once the signal starts, the Page test statistic does not reset to zero before crossing the threshold. Let $\mathcal{A}_m$ for $m < 0$ represent the event that $W_m = 0$ and that $W_{m+1}, W_{m+2}, \ldots, W_0$ are not equal to zero. Then for $m < 0$ the PMF of M is

$$
p_M[m] = \Pr\{\mathcal{S}(\mathbf{u}_0)\,|\mathcal{A}_m\}, \qquad (18)
$$

where $\mathbf{u}_0 = \begin{bmatrix} 0 & \mathbf{v}_0^T \end{bmatrix}^T$ is the state probability vector at time $k = 0$. The reduced state probability vector at time $k = 0$ ($\mathbf{v}_0$) is now determined subject to the occurrence of event $\mathcal{A}_m$. The transition of the states from time $k-1$ to k for $m+1 < k \leq 0$ may be represented by

$$
\mathbf{v}_k = \mathbf{D}_0\mathbf{v}_{k-1}, \qquad (19)
$$

where $\mathbf{D}_0$ is a $\gamma - 1$ by $\gamma - 1$ matrix of the probability of a transition from one state to another under H_0,

$$
\mathbf{D}_0 = \begin{bmatrix} p_0 & p_{-1} & \cdots & p_{-\gamma+2} \\ p_1 & p_0 & \cdots & p_{-\gamma+3} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\gamma-2} & p_{\gamma-3} & \cdots & p_0 \end{bmatrix}. \qquad (20)
$$

Since event $\mathcal{A}_m$ requires that $W_m = 0$ and $W_{m+1} \neq 0$, the transition of the states from time m to time $m+1$ is restricted to moving from state l_0 to the states $l_1, \ldots, l_{\gamma-1}$ under H_0. Thus, the reduced state probability vector for time $k = m+1$ is

$$\mathbf{v}_{m+1} = \left(\mathbf{e}_0^T \mathbf{u}_{ss}\right) \mathbf{d}_0, \tag{21}$$

where

$$\mathbf{d}_0 = \left[p_1 \; p_2 \; \cdots \; p_{\gamma-1}\right]^T, \tag{22}$$

$\mathbf{e}_0^T \mathbf{u}_{ss}$ is the steady state probability of observing state l_0, and $\mathbf{u}_{ss}$ is the steady state probability of observing the states. As derived in reference [5], $\mathbf{u}_{ss}$ is the eigenvector (normalized to sum to one) associated with the maximum eigenvalue of the probability transition matrix including regulation at zero under H_0,

$$\mathbf{C}_0 = \begin{bmatrix} \sum_{i=-\infty}^{0} p_i & \sum_{i=-\infty}^{-1} p_i & \cdots & \sum_{i=-\infty}^{-(\gamma-1)} p_i \\ p_1 & p_0 & \cdots & p_{-\gamma+2} \\ p_2 & p_1 & \cdots & p_{-\gamma+3} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\gamma-1} & p_{\gamma-2} & \cdots & p_0 \end{bmatrix}. \tag{23}$$

Now, the reduced state probability vector for time $k = 0$ subject to the occurrence of event $\mathcal{A}_m$ may be determined by combining Eqs. (19) and (21),

$$\mathbf{v}_0 = \left(\mathbf{e}_0^T \mathbf{u}_{ss}\right) \mathbf{D}_0^{-m-1} \mathbf{d}_0. \tag{24}$$

Now consider case (ii) where, once the signal starts, the Page test statistic resets to zero before crossing the threshold. Let $\mathcal{B}_m$ for $m \geq 0$ represent the event that $W_m = 0$ and that at time $k = 0$ the Page test statistic is in a steady state condition (i.e., the state probability vector is $\mathbf{u}_{ss}$). Then for $m \geq 0$ the PMF of M is

$$p_M[m] = \Pr\{\mathcal{S}(\mathbf{e}_0)\} \Pr\{\mathcal{B}_m\}. \tag{25}$$

Let $\mathbf{u}_k$ be the state probability vectors for $k = 0, 1, \ldots, m$. As described in reference [5], they are related by the product

$$\mathbf{u}_k = \mathbf{C}_1 \mathbf{u}_{k-1}, \tag{26}$$

where $\mathbf{C}_1$ is the probability transition matrix including regulation at zero under H_1,

$$\mathbf{C}_1 = \begin{bmatrix} \sum_{i=-\infty}^{0} q_i & \sum_{i=-\infty}^{-1} q_i & \cdots & \sum_{i=-\infty}^{-(\gamma-1)} q_i \\ q_1 & q_0 & \cdots & q_{-\gamma+2} \\ q_2 & q_1 & \cdots & q_{-\gamma+3} \\ \vdots & \vdots & \ddots & \vdots \\ q_{\gamma-1} & q_{\gamma-2} & \cdots & q_0 \end{bmatrix}. \tag{27}$$

The probability of observing state l_0 at time m when the Page test is in its steady state at time $k = 0$ is then

$$\begin{aligned} \Pr\{\mathcal{B}_m\} &= \mathbf{e}_0^T \mathbf{u}_m \\ &= \mathbf{e}_0^T \mathbf{C}_1^m \mathbf{u}_{ss}. \end{aligned} \tag{28}$$

Combining Eqs. (15), (18), (24), (25) and (28) results in the following definition for the probability mass function of the error of the stopping time estimator of Eq. (2),

$$p_M[m] = \begin{cases} \left(\mathbf{e}_0^T \mathbf{u}_{ss}\right) \mathbf{w}^T \mathbf{P}^T \mathbf{D}_0^{-m-1} \mathbf{d}_0 & m < 0 \\ \left(\mathbf{e}_0^T \mathbf{w}\right) \mathbf{e}_0^T \mathbf{C}_1^m \mathbf{u}_{ss} & m \geq 0 \end{cases}. \tag{29}$$

2.3 First three moments

Define the i^{th} uncorrected moment of the error of the starting time estimator as

$$\begin{aligned} \mu_i &= E\left[M^i\right] \\ &= (-1)^i \left(\mathbf{e}_0^T \mathbf{u}_{ss}\right) \mathbf{w}^T \mathbf{P}^T \left(\sum_{m=1}^{\infty} m^i \mathbf{D}_0^{m-1}\right) \mathbf{d}_0 \\ &\quad + \left(\mathbf{e}_0^T \mathbf{w}\right) \mathbf{e}_0^T \left(\sum_{m=0}^{\infty} m^i \mathbf{C}_1^m\right) \mathbf{u}_{ss}. \end{aligned} \tag{30}$$

Evaluation of Eq. (30) for $i = 0$, 1, 2, and 3 is straightforward using closed forms for the infinite summations as found in reference [6]. Note that this requires that the eigenvalue with the maximum absolute value of the matrices $\mathbf{D}_0$ and $\mathbf{C}_1$ be less than one. With the use of Perron's theorem [7], which dictates that all the eigenvalues lie on a disk of radius equal to the maximum absolute column sum, it suffices to note that all the entries in the matrices $\mathbf{D}_0$, $\mathbf{D}_1$ (from Sect. 2.2), and $\mathbf{C}_1$ are non-negative and that the maximum (absolute) column sum is always less than one[1]. Equation (30) then results in the forms

$$\mu_0 = \left(\mathbf{e}_0^T \mathbf{u}_{ss}\right) \mathbf{w}^T \mathbf{P}^T \left(\mathbf{I}_{\gamma-1} - \mathbf{D}_0\right)^{-1} \mathbf{d}_0 + \left(\mathbf{e}_0^T \mathbf{w}\right) \mathbf{e}_0^T \left(\mathbf{I}_\gamma - \mathbf{C}_1\right)^{-1} \mathbf{u}_{ss}, \tag{31}$$

$$\mu_1 = -\left(\mathbf{e}_0^T \mathbf{u}_{ss}\right) \mathbf{w}^T \mathbf{P}^T \left(\mathbf{I}_{\gamma-1} - \mathbf{D}_0\right)^{-2} \mathbf{d}_0 + \left(\mathbf{e}_0^T \mathbf{w}\right) \mathbf{e}_0^T \left(\mathbf{I}_\gamma - \mathbf{C}_1\right)^{-2} \mathbf{C}_1 \mathbf{u}_{ss}, \tag{32}$$

$$\begin{aligned} \mu_2 &= \left(\mathbf{e}_0^T \mathbf{u}_{ss}\right) \mathbf{w}^T \mathbf{P}^T \left(\mathbf{I}_{\gamma-1} + \mathbf{D}_0\right) \left(\mathbf{I}_{\gamma-1} - \mathbf{D}_0\right)^{-3} \mathbf{d}_0 \\ &\quad + \left(\mathbf{e}_0^T \mathbf{w}\right) \mathbf{e}_0^T \left(\mathbf{I}_\gamma + \mathbf{C}_1\right) \left(\mathbf{I}_\gamma - \mathbf{C}_1\right)^{-3} \mathbf{C}_1 \mathbf{u}_{ss}, \end{aligned} \tag{33}$$

[1] With the exception of degenerate cases where $g(X)$ does not have support over certain regions of the real line.

and

$$\mu_3 = -\left(\mathbf{e}_0^T \mathbf{u}_{ss}\right) \mathbf{w}^T \mathbf{P}^T \left(\mathbf{I}_{\gamma-1} + 4\mathbf{D}_0 + \mathbf{D}_0^2\right) \left(\mathbf{I}_{\gamma-1} - \mathbf{D}_0\right)^{-4} \mathbf{d}_0 + \left(\mathbf{e}_0^T \mathbf{w}\right) \mathbf{e}_0^T \left(\mathbf{I}_\gamma + 4\mathbf{C}_1 + \mathbf{C}_1^2\right) \left(\mathbf{I}_\gamma - \mathbf{C}_1\right)^{-4} \mathbf{C}_1 \mathbf{u}_{ss}. \tag{34}$$

Note that Eq. (31) should result in $\mu_0 = 1$.

3

Verification of analysis

The PMF of the starting time estimator derived in Sect. 2 is verified by comparing the analytical distribution to one estimated through simulation. A Gaussian shift-in-mean signal is considered where under H_0 the observed data are zero-mean unit-variance Gaussian random variables and under H_1 the mean changes to the positive value μ. The log-likelihood ratio is used as the detector nonlinearity,

$$g(x) = \mu \left(x - \frac{\mu}{2} \right). \tag{35}$$

In the simulation 10,000 trials are run with a threshold $h = 10$ where the actual signal start time is uniformly distributed between time samples 101 and 110 to ensure that the Page test is in its steady state before the change occurs. From the trials, a sample PMF and CDF of the error of the starting time estimator of Eq. (2) are generated and compared to that computed from Eq. (29) using 100 quantization levels as shown respectively in Figs. 1 and 2. Visually there is little difference between the estimated and theoretical PMFs and CDFs.

To further substantiate the validity of the analytical result of Eq. (29) and to indicate how fine the quantization needs to be, the p-value of the Kolmogorov-Smirnov statistic is shown as a function of the number of quantization levels in Fig. 3. The p-value indicates the probability of exceeding the observed value of a random variable. Thus, if the p-value is small, there is a significant difference between the observed and estimated CDF. As expected, the difference becomes significant as the number of quantization levels decreases. For this example (i.e., signal type, strength and detector threshold), the approximation becomes suspect when the number of quantization steps falls below 100. The p-value is calculated using the asymptotic distribution of the Kolmogorov-Smirnov statistic as described in Fisz [8].

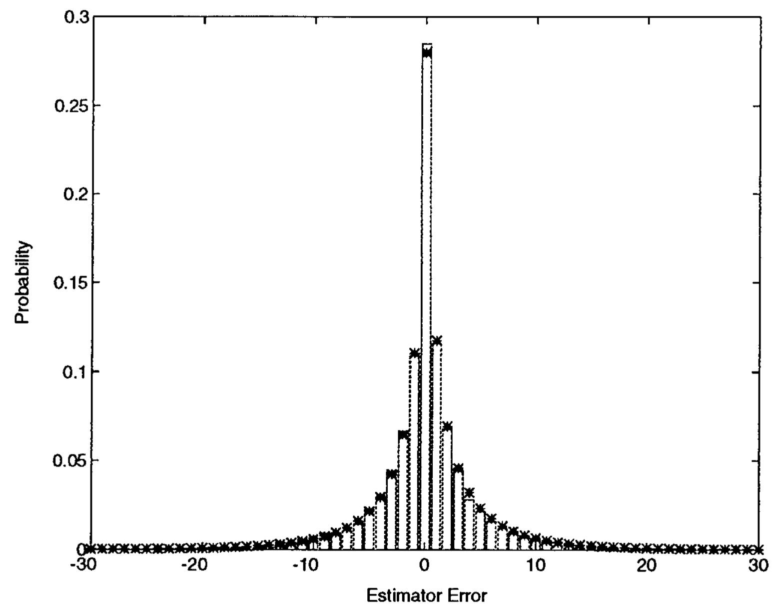

Figure 1: Comparison of analytical (*) and estimated (bar graph) PMFs of starting time estimator error.

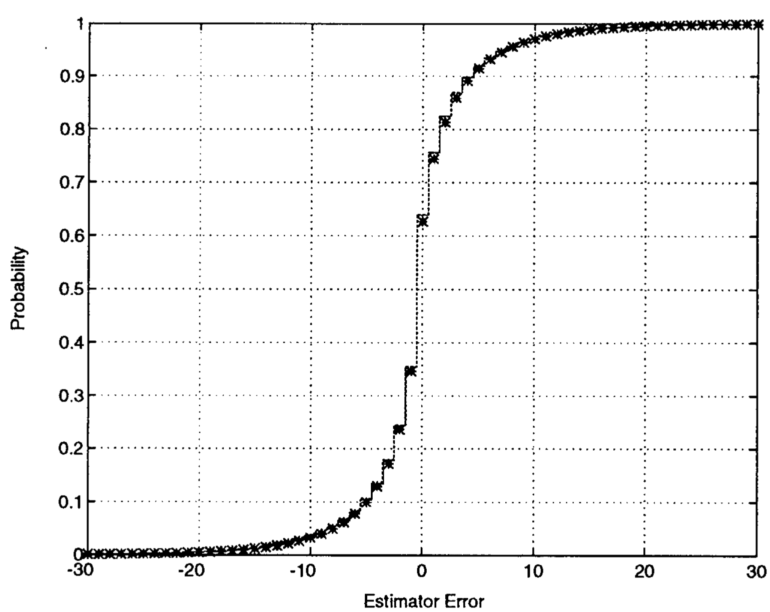

Figure 2: Comparison of analytical (*) and estimated (staircase) CDFs of starting time estimator error.

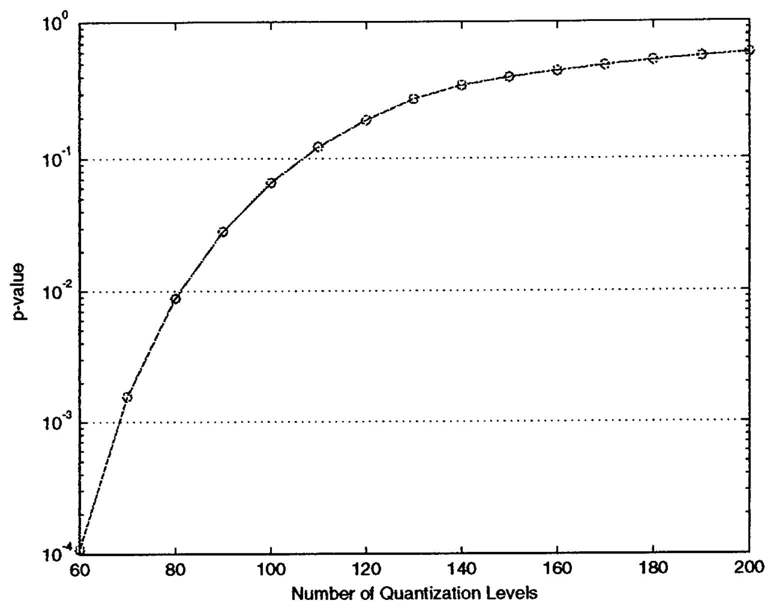

Figure 3: p-value of the Kolmogorov-Smirnov statistic versus the number of quantization levels for Gaussian shift-in-mean signal.

4

Estimator performance

The performance of the estimator in terms of the bias, standard deviation and skewness are examined as a function of signal strength for Gaussian shift-in-mean and noncentral chi-squared signal types. The bias is the first uncorrected moment (μ_1), the standard deviation is,

$$\sigma = \sqrt{\mu_2 - \mu_1^2}, \tag{36}$$

and the skewness is

$$\kappa = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{\sigma^3}. \tag{37}$$

4.1 Matched detector nonlinearity

The detector nonlinearity for the Gaussian shift-in-mean signal type is the log-likelihood ratio as described in Eq. (35). For the latter signal type, the data are centrally chi-squared distributed under H_0 ($X_i \sim \mathcal{X}_n^2$) and noncentrally chi-squared distributed under H_1 with a noncentrality parameter of δ ($X_i \sim \mathcal{X}_n^2(\delta)$) and degrees of freedom n. The locally optimal detector nonlinearity is used with the asymptotically optimal bias [9]

$$g(x) = x - \tau_\delta, \tag{38}$$

where

$$\tau_\delta = n\left(1 + \frac{n}{\delta}\right)\log\left(1 + \frac{\delta}{n}\right). \tag{39}$$

In both cases, the detector nonlinearity is *matched* to the actual signal strength; that is, the actual signal strength is known and used to form the detector nonlinearity. Define the signal power for the Gaussian shift-in-mean signal type as $\delta = \mu^2$ and simply δ for the noncentral chi-squared signal type. The degrees of freedom parameter for the noncentral chi-squared signal is set to $n = 2$.

The bias, standard deviation, and skewness of the starting time estimator for both signal types are respectively found in Figs. 4, 5, and 6 as a function of signal strength. As expected, both the bias and variance decrease as signal strength increases. Additionally, the bias for the noncentral chi-squared signal is substantially larger than the bias for the Gaussian shift-in-mean signal, particularly at lower signal strengths.

4.2 Mismatched detector nonlinearity

In most applications, the signal strength is not known exactly. In these situations a design signal strength is chosen (say $\tilde{\mu}$ and $\tilde{\delta}$) and substituted into the detector nonlinearity. The bias, standard deviation and skewness for a design signal strength $\tilde{\mu}^2 = \tilde{\delta} = 1$ are respectively found in Figs. 4, 5, and 6 for the Gaussian shift-in-mean and noncentral chi-squared signal types. Here it is seen that the bias and variance increase as the signal strength decreases at a greater rate than if the detector nonlinearity were exactly matched to the signal strength, particularly for the Gaussian shift-in-mean signal.

When the detector nonlinearity is formed based on a design signal strength, there is a minimum detectable signal level. For the Page test, this is usually chosen as the signal strength that causes the data transformed by the detector nonlinearity to have zero mean[2]. With a 0 db design signal strength, this occurs at approximately -3 db for the Gaussian shift-in-mean signal and at about -3.6 dB for the noncentral chi-squared signal. In these regions it is clear that the performance of the starting time estimator is unacceptable.

The bias and skewness both exhibit negative values for high signal strengths whereas when the detector nonlinearity is exactly matched to the observed signal strength the bias and skewness are always positive. The negative bias and skewness values occur because the probability of crossing the threshold immediately after the signal starts without a reset to zero is large. Thus, the starting time estimator will often have negative error values (which is not necessarily bad).

4.3 Detector threshold

The starting time estimator of Eq. (2) is based on the Page test statistic. Thus, the design parameters of the Page test may affect the performance of the proposed starting time estimator. The primary design parameter of the Page test is the threshold used to declare signal-presence; that is, the value that the Page test statistic must exceed for a detection to occur. The detector threshold is usually chosen according to constraints on the false alarm performance, with large thresholds corresponding to large average times between false alarms. An increase in the Page test detector threshold results in a decrease in the probability of detecting finite duration signals and an increase in the average delay before detection. The effect of varying the Page test detector threshold on the bias, standard deviation, and skewness of the proposed starting time estimator for 0 dB Gaussian shift-in-mean and noncentral chi-squared signals with a matched detector nonlinearity is found in Figs. 7, 8, and

[2]Because the average number of samples before detection becomes approximately exponentially related to the threshold for negative means as opposed to linearly related for positive means.

9. The only expected trend that may be observed is the increase in variance as the threshold increases. Thus, it may be desirable to have a lower threshold to reduce the variance on the signal starting time estimator; however, the requirements on the false alarm performance of the detector may take precedence and counter this desire. It should also be noted that for the Gaussian shift-in-mean signal the performance does not vary substantially with the threshold.

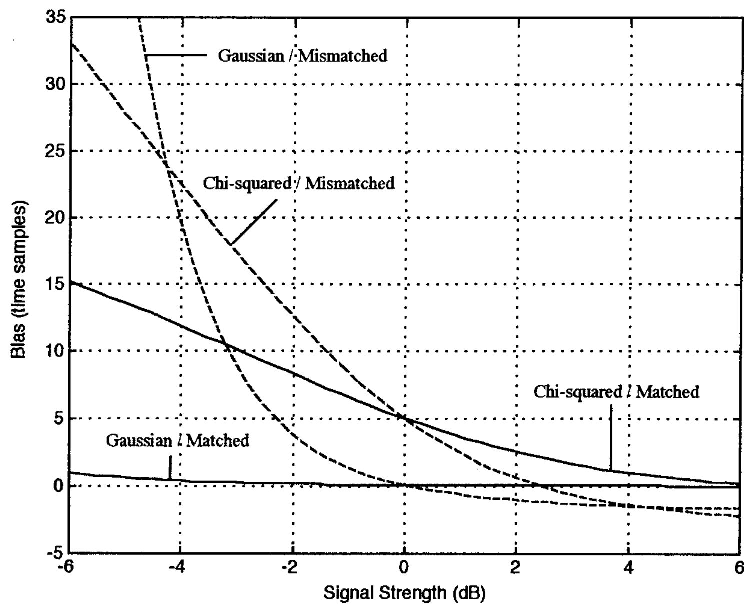

Figure 4: Bias for the matched and mismatched detector nonlinearities for Gaussian shift-in-mean and noncentral chi-squared signal types.

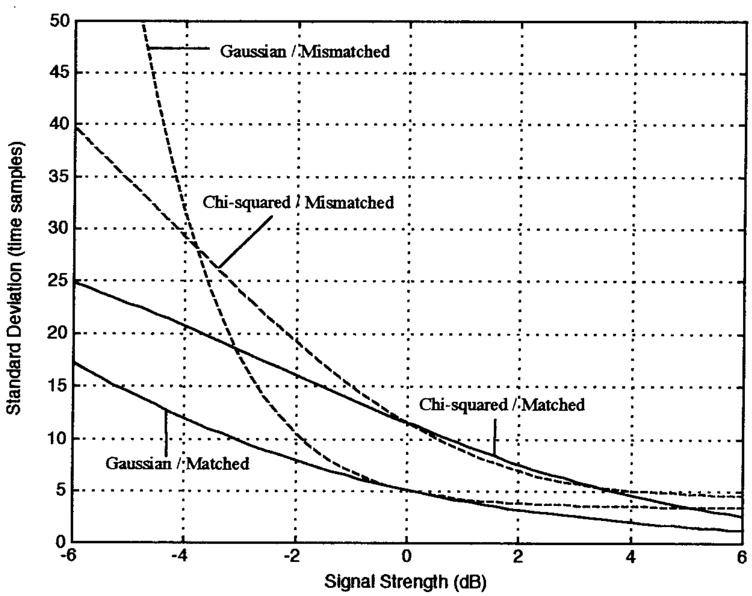

Figure 5: Standard deviation for the matched and mismatched detector nonlinearities for Gaussian shift-in-mean and noncentral chi-squared signal types.

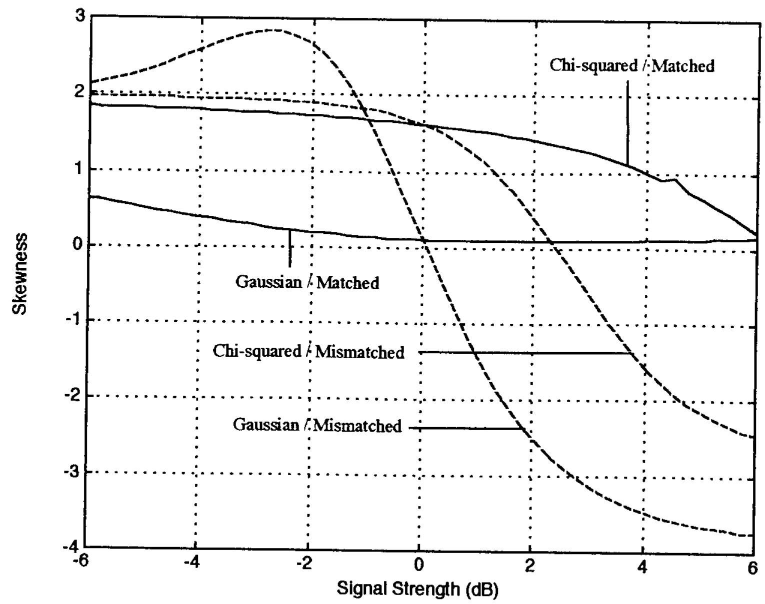

Figure 6: Skewness for the matched and mismatched detector nonlinearities for Gaussian shift-in-mean and noncentral chi-squared signal types.

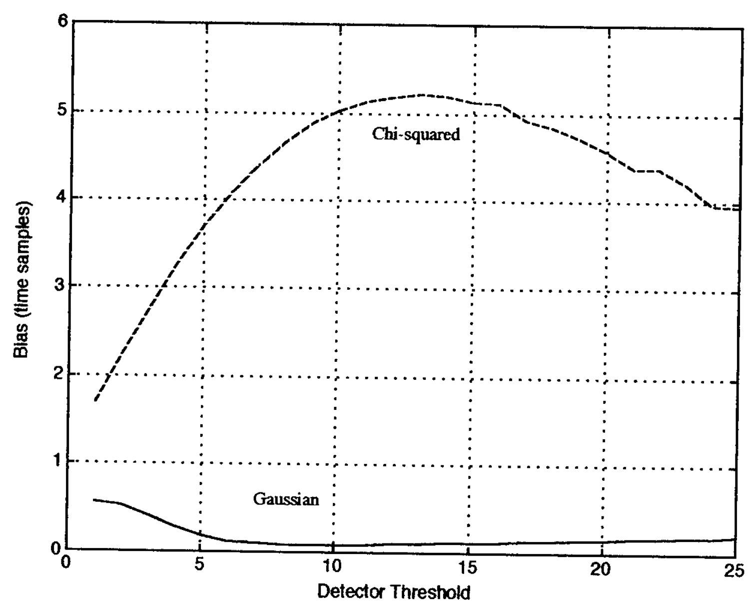

Figure 7: Bias for 0 dB signal with the matched detector nonlinearity for Gaussian shift-in-mean and noncentral chi-squared signal types.

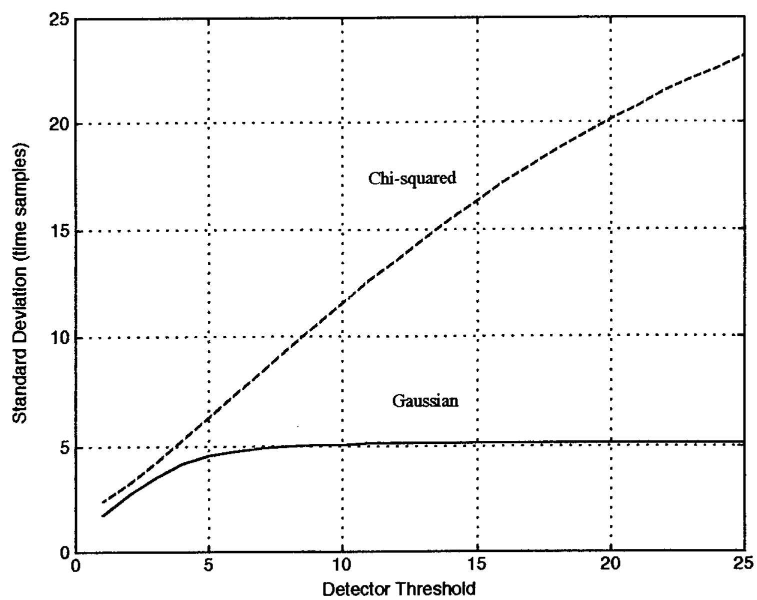

Figure 8: Standard deviation for 0 dB signal with the matched detector nonlinearity for Gaussian shift-in-mean and noncentral chi-squared signal types.

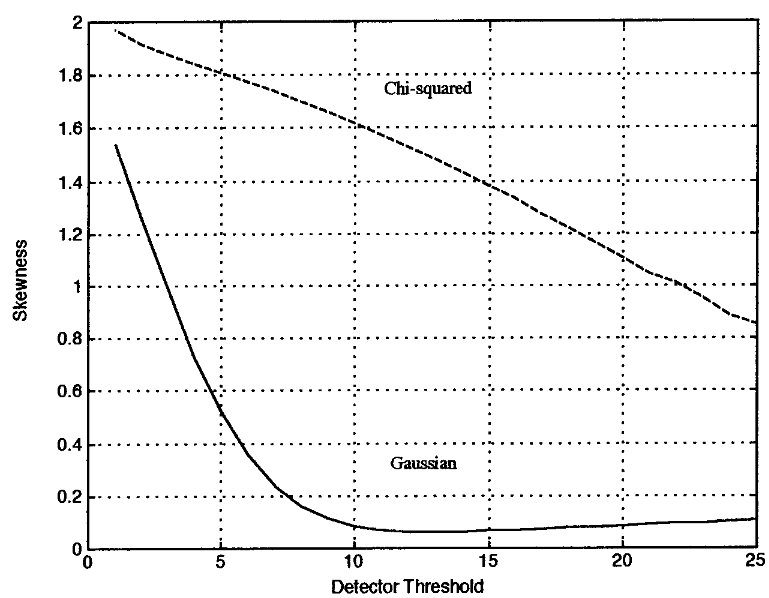

Figure 9: Skewness for 0 dB signal with the matched detector nonlinearity for Gaussian shift-in-mean and noncentral chi-squared signal types.

4.4 Upper quantile

The objective in obtaining an estimate of the signal starting and stopping time is usually to perform further processing on the segment of data that contains the signal. It may be desirable to ensure, within some probability, that the starting time estimate is less than or equal to the actual starting time; that is, force a negative bias on the starting time estimator. For instance, the upper α-quantile may be subtracted from the starting time estimator so that the probability of observing a positive error is small (α). This provides the desired negative bias and indicates how often the starting time estimator misses the actual start of the signal. The value to be subtracted is the positive integer-valued quantile z_α where

$$\begin{aligned} \alpha &\leq \Pr\{M \geq z_\alpha\} \\ &= \sum_{m=z_\alpha}^{\infty} p_M[m] \\ &= \left(\mathbf{e}_0^T \mathbf{w}\right) \mathbf{e}_0^T (\mathbf{I}_\gamma - \mathbf{C}_1)^{-1} \mathbf{C}_1^{z_\alpha} \mathbf{u}_{ss}. \end{aligned} \tag{40}$$

The value of $z_{0.01}$ for both signal types and the matched and mismatched detector nonlinearities is shown as a function of signal strength in Fig. 10. The mismatched detector nonlinearities are designed assuming a 0 dB signal strength and the detector threshold was set at $h = 10$.

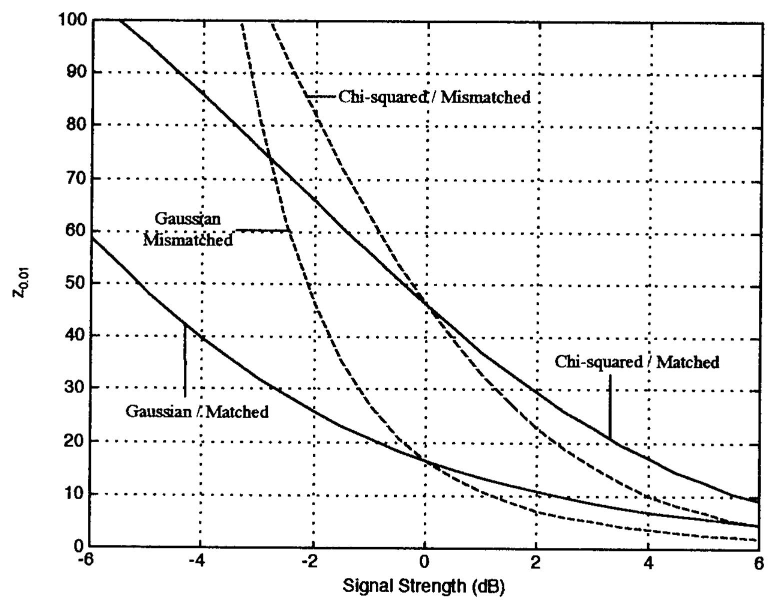

Figure 10: $z_{0.01}$ for Gaussian shift-in-mean and noncentral chi-squared signal types.

5 Summary

Many signal processing applications require the detection of an abrupt change with subsequent estimation of the actual time of occurrence of the change. A starting time estimator based on the Page test statistic has been proposed for this purpose. The PMF and first three uncorrected moments of the estimator error have been determined analytically subject to a quantization of the Page test statistic update. The performance of the estimator for Gaussian shift-in-mean and noncentral chi-squared signal types was evaluated revealing and quantifying the expected improvement in performance (i.e., decrease in bias and variance) as signal strength increased.

In many applications the initial segment of the signal may contain information crucial to the post-processing. In these situations it is desirable to induce a negative bias on the starting time estimator to ensure, within some probability, that the estimator captures the start of the signal (i.e., the estimate is less than or equal to the actual starting time with some given probability). Thus, the upper tail probability of the error of the starting time estimator was derived from the PMF and used to determine the induced bias providing a given probability of capturing the start of the signal for the Gaussian shift-in-mean and noncentral chi-squared signal types.

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Document Data Sheet

NATO UNCLASSIFIED

Security Classification		Project No.
NATO UNCLASSIFIED		20
Document Serial No.	**Date of Issue**	**Total Pages**
SM-295	December 1995	27 pp.

Author(s)

D.A. Abraham

Title

Analysis of a signal starting time estimator based on the Page test statistic.

Abstract

The time of the most recent reset to zero of the Page test statistic is proposed as an estimator of the starting time of a signal. The probability mass function of the estimator is determined analytically subject to a quantization of the Page test statistic update. Closed form results for the first three uncorrected moments of the estimator are presented. The analytical results are verified by comparison with simulation results and the fineness of the quantization required for accurate representation is investigated by evaluation of the Kolmogorov-Smirnov statistic. The bias, standard deviation, and skewness of the estimator as a function of the signal strength and detector threshold are evaluated for Gaussian shift-in-mean and noncentral chi-squared signal types.

Keywords

Page test – submarine detection, localization, classification – signal starting time estimator

Issuing Organization

North Atlantic Treaty Organization
SACLANT Undersea Research Centre
Viale San Bartolomeo 400, 19138 La Spezia, Italy

[From N. America: SACLANTCEN CMR-426 (New York) APO AE 09613]

Tel: +39 (0)187 540 111
Fax:+39 (0)187 524 600

E-mail: library@saclantc.nato.int

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